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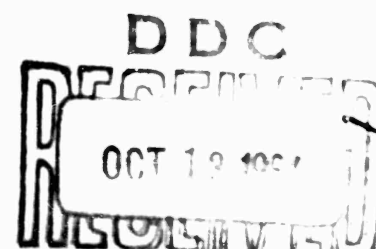
SOME THEOREMS CONCERNING THE MOTION OF AN
ELECTRICALLY CHARGED PARTICLE IN A DIPOLE MAGNETIC FIELD

Ernest C. Ray

December 1961

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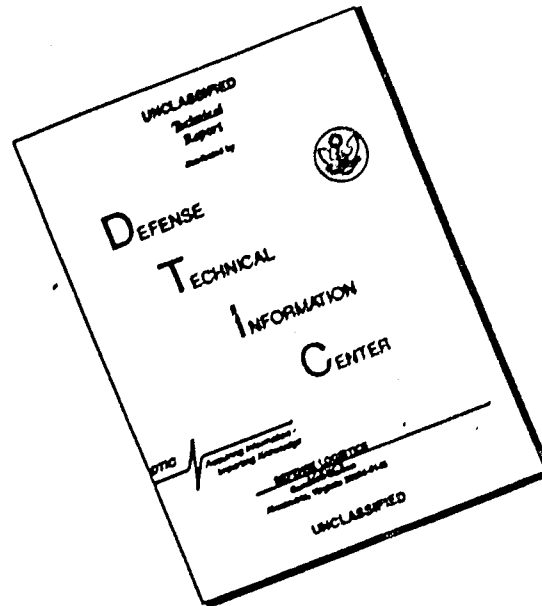
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PREFACE

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SOME THEOREMS CONCERNING THE MOTION OF AN
ELECTRICALLY CHARGED PARTICLE IN A DIPOLE MAGNETIC FIELD

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CHARGED PARTICLE IN MAGNETIC FIELD

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ABSTRACT

We prove various theorems related to the application to cosmic rays of the theory of the motion of an electrically charged particle in a dipole magnetic field. The theorems are essentially those conjectured by Schremp. In making the proofs, we assume that the so called trajectories of the first and second kinds have certain properties in the large. These properties can be verified numerically and by series expansions in any particular case.

INTRODUCTION

The theory of the motion of an electrically charged particle in a dipole magnetic field was initiated by Störmer (1) and extended by Lemaitre and Vallarta. (2) The notation used in the present work will be that of Vallarta, except that his γ_1 will be called simply γ . Recently, it has been found that the higher multipole terms of the earth's magnetic field play a sufficiently important role in determining the behavior of cosmic rays that no quantitatively adequate theory of cut-offs can be based on the dipole model. Nevertheless, we feel that the dipole theory warrants the present study on two grounds. In the first place, the dipole theory furnishes most of the available insight into the cosmic ray problem. In the second place, it is an example of a class of very interesting problems in classical mechanics which do not submit to the usual text book methods, and is therefore of interest for its own sake.

Störmer found a necessary condition that a particle must satisfy in order to travel from infinity so as to arrive at the earth's surface at a given geomagnetic latitude from a given direction. His condition is that the rigidity of the particle must exceed a certain critical value which depends on latitude and direction of arrival.

Lemaitre (3) found a sufficient condition of the same sort. He showed that for γ very near unity, the trajectories asymptotic to the outer periodic orbit serve as boundaries of sets of trajectories all of which join an observation point on the earth with infinity without passing through the earth between these two points. He conjectured this same property to hold for all values of γ for which the asymptotic trajectories exist. He and Vallarta calculated families of them on Bush's differential

analyzer. In addition, Lemaitre found an additional necessary condition for arrival which he calls the shadow cone. Finally, Schremp (4) conjectured, from an inspection of the trajectories produced by the differential analyzer, that for particles with rigidities greater than a certain critical value, Lemaitre's sufficient condition is necessary as well. We present proofs of these claims, slightly extend them, and clarify the relations between them.

THE EQUATIONS OF MOTION

The equations of motion are obtained from the Lorentz force. They are thoroughly discussed by Störmer. (1) The problem has three degrees of freedom and two known first integrals. One of these is the energy, which in the present case reduces to the constancy of the particle speed, even in the correct relativistic case. Because of this integral, the correct relativistic equations have the same form as the non-relativistic ones, provided m , the particle mass, is taken to be $m_0(1 - v^2/c^2)^{-1/2}$ with m_0 the rest mass and c the speed of light. The second first integral arises from the axial symmetry of the problem and expresses the conservation of the azimuthal component of the angular momentum.

Let r denote distance from the dipole, λ the geomagnetic latitude, and s path length along a trajectory. Put

$$e^x = 2\gamma r, \quad d\sigma = 2\gamma r^{-2} ds \quad (1)$$

where γ is the constant of angular momentum.

Störmer obtains the equations of motion in the form

$$\frac{d^2x}{d\sigma^2} = ae^{2x} - e^{-x} + e^{-2x} \cos^2\lambda \quad (2.a)$$

$$\frac{d^2\lambda}{d\sigma^2} = e^{-2x} \sin\lambda \cos\lambda - \sin\lambda \cos^3\lambda. \quad (2.b)$$

We have put $a = (2\gamma)^{-4}$ and the angular momentum integral has been used to reduce the equations. All lengths are in Störmer units. That is, the unit of length is $(M/R)^{1/2}$ where M is the dipole moment and R the particle rigidity. The energy integral, which has not yet been used, is

$$\left(\frac{dx}{d\sigma}\right)^2 + \left(\frac{d\lambda}{d\sigma}\right)^2 = ae^{2x} + 2e^{-x} - e^{-2x} \cos^2\lambda - \cos^2\lambda. \quad (3)$$

By introducing two new variables x' and λ' we may write Eqs. (2)

as

$$\frac{dx'}{d\sigma} = ae^{2x} - e^{-x} + e^{-2x} \cos^2\lambda \quad (4.a)$$

$$\frac{dx}{d\sigma} = x' \quad (4.b)$$

$$\frac{d\lambda'}{d\sigma} = e^{-2x} \sin\lambda \cos\lambda - \sin\lambda \cos^3\lambda \quad (4.c)$$

$$\frac{d\lambda}{d\sigma} = \lambda'. \quad (4.d)$$

The energy integral is then the purely algebraic relation obtained by substituting from (4.b) and (4.d) in (3).

An alternative formulation is obtained by using the energy integral to reduce the system of Eqs. (4). We obtain

$$\frac{dx}{d\sigma} = \pm f^{1/2}(x, \lambda, \lambda'; \gamma) \quad (5.a)$$

$$\frac{d\lambda'}{d\sigma} = g(x, \lambda) \quad (5.b)$$

$$\frac{d\lambda}{d\sigma} = \lambda' \quad (5.c)$$

where

$$f(x, \lambda, \lambda'; \gamma) \equiv ae^{2x} + 2e^{-x} - e^{-2x} \cos^2 \lambda - \cos^{-2} \lambda - \lambda'^2$$

and

$$g(x, \lambda) \equiv e^{-2x} \sin \lambda \cos \lambda - \sin \lambda \cos^{-3} \lambda.$$

LOCAL PROPERTIES OF SOLUTIONS

Throughout the paper we consider only real solutions for real values of σ . Consider first solutions of (4). The right members have bounded partial derivatives except at $\lambda = \pi/2$ and $x = \pm \infty$. Therefore, except at these places, for a given value of γ exactly one solution passes through each point of the space $(x, \lambda, x', \lambda')$. It is important to notice that this is still true when $x' = 0$.

The only solutions in which we are interested are those for which the energy integral is (3) without the addition of a new constant. These are obviously the set of all solutions of (5). Every solution of (5) is a solution of (4). Every solution of (4) for which the energy constant is chosen as in (3) is a solution of (5). Thus, through every point of the space (x, λ, λ') such that $f(x, \lambda, \lambda'; \gamma) > 0$ pass two and only two solutions of (5), corresponding to the two choices of sign in (5.a). Through every point for which $f(x, \lambda, \lambda'; \gamma) = 0$ passes exactly one solution. When $f < 0$, $dx/d\sigma$ is imaginary, and no real solutions occur. The surface $f(x, \lambda, \lambda'; \gamma) = 0$ is the boundary separating regions in which motions occur from those in which it does not. We will call it the motion boundary. On it, $dx/d\sigma = 0$, elsewhere $dx/d\sigma \neq 0$. Consequently every trajectory has all of its extrema in x on the motion boundary. This fact is of central importance for all that follows.

Next, it is necessary to understand the structure of the motion boundary. It is evident that its intersection with the plane $\lambda' = 0$ is the boundary of Störmer's forbidden regions. We shall only be interested in the cases $0 < \gamma < 1$. For every such case, the Störmer boundaries have the structure shown in Fig. 1. The shaded regions are forbidden. It is easy to see from the algebraic equation $f(x, \lambda, \lambda'; \gamma) = 0$ that when $0 < \gamma < 1$ the motion boundary has the structure of a "Y" constructed of hollow tubing.

SPECIAL SOLUTIONS

The equations (5) are not integrable. All of our knowledge of specific trajectories depends on the application of numerical methods.

Störmer discovered certain periodic solutions which are now called principal periodic orbits. We will call these simply "principal orbits." For a particular value of γ , which we will call γ^* , there is one of these. If $\gamma^* < \gamma \leq 1$ there are two. Godart (5) has calculated γ^* to six places as $\gamma^* = 0.788541$. These orbits were studied extensively by Lemaitre, Vallarta, and their students. Godart initiated a study of the stability of them. The principal orbit which is farther from the dipole (called the outer orbit) is unstable. The inner orbit is unstable for certain ranges of γ . Over the remaining range of γ , the inner orbit has imaginary characteristic exponents (6) so that its stability is undecided. In this paper we will be concerned only with the outer orbit.

Since the outer orbit is unstable, we know from Poincaré that there are orbits asymptotic to it. These have been extensively calculated by Lemaitre and Vallarta, who made them the foundation of their theory of the allowed cone. (2) Figure 2 shows a sampling of the family asymptotic from the side toward the dipole, as calculated by them. The calculational

methods which they developed are adequate to verify the properties of asymptotic trajectories in which we shall be interested in this paper, and we will not go into them here. We will instead assume that the family has particular properties of interest and base our proof on these assumptions. In a practical case it would then be necessary to carry out the procedure that they specify (7) in order to find the regions of space in which the theorem is applicable. This is no hardship since the theorem cannot in any case be put to use without the asymptotic families.

THE MAIN CONE

In this section we prove a theorem which, under appropriate circumstances to be specified, justifies the use by Lemaitre and Vallarta of the asymptotic trajectories as boundaries of allowed cones of cosmic radiation. Before stating the theorem it is necessary to discuss some properties of asymptotic solutions as plotted in the (x, λ, λ') space.

We begin by considering the trajectories plotted in the (x, λ) plane. See Fig. 2. The slope in such a plot is $d\lambda/dx = \lambda'/x'$. From Eq. (3)

$$\left[1 + (\lambda'/x')^{-2}\right] (\lambda')^2 = ae^{2x} + 2e^{-x} - e^{-2x} \cos^2 \lambda - \cos^2 \lambda.$$

Thus when one has definite values of x , λ , a , and $d\lambda/dx$, one has a value of λ' which is determined up to a sign. If, in addition, one has a definite sign for x' , one has a definite sign for λ' . Finally, from Eq. (3) directly, for x , λ and a fixed. λ' and x' vary in opposite directions so that $d\lambda/dx$ and λ' vary in the same direction. This fact makes it easy to compare the relative size of λ' on each of two trajectories passing through a given point.

Let us now see how the asymptotic trajectories lie in the (x, λ, λ') space. First consider the periodic orbit. In the (x, λ) plane a particle

travels back and forth along it from one end to the other. It is fairly obvious that when plotted in x, λ, λ' space it becomes a simple closed curve. Since at any point on it when $x' = 0$ it is tangent to the motion boundary, we see from Fig. 2 that it makes contact with that boundary in four points. These points are marked with open circles on the figure. The point in the center counts as two, of course. On one-half of the orbit $\lambda' > 0$, on the other half $\lambda' < 0$, so that the plane $\lambda' = 0$ bisects the orbit. In fact, it is a plane of symmetry for it. On the other hand, x' alternates in sign between any two adjacent quadrants. Now, if we add to the figure a λ' axis at right angles to the figure and with the positive λ' direction out of the page, we can conveniently visualize this periodic orbit as so plotted in the resulting 3-space.

Now we come to the asymptotic trajectories themselves. Let us take them as being traversed by the particle in the direction shown by the arrows, so that the particle approaches the periodic orbit in the infinite future.

It has been shown by Poincaré that such a family of asymptotic trajectories constitute a surface, and that the periodic orbit lies on the same surface. (8) Lemaitre and Vallarta have adapted the method of proof used by Poincaré to the calculation of the family. (7) One can "see" this surface in Fig. 2. If one visualizes this plot as it would be in x, λ, λ' space, the asymptotic trajectories seem to wind spirally around a "cylinder." With Poincaré, we will call this surface the asymptotic surface. A way of labeling points on this surface that is sometimes convenient arises from Poincaré's proof. The asymptotic trajectories are distinguished from each other by a single initial condition, which can be used as one label. The arc length σ on the asymptotic trajectory selected serves as the other.

We now describe the property of a family of segments of asymptotic trajectories on which our proof of Lemaitre's main cone theorem depends. Consider a particular plane $x = \text{constant}$. In the geophysical application, the earth's surface is such a plane. We will therefore call this plane the earth. The vertical line in Fig. 2 labelled x_1 is such a plane, and has the property to be described. We will call the complete set of asymptotic trajectories the asymptotic surface. We assume that

1. The earth intersects the asymptotic surface.
2. At least part of the intersection (the only part we will consider) consists of a closed curve which bounds a simply connected region on the earth.
3. Each asymptotic trajectory has, at this intersection, $x' > 0$.
4. No asymptotic trajectory, on leaving this intersection, subsequently intersects the earth.

The only part of the asymptotic surface which we subsequently consider is that composed of trajectory segments leaving this intersection and proceeding into the infinite future to become asymptotic to the outer orbit, and one of the two families asymptotic on the outer side of the outer orbit.

Notice that there are portions of this asymptotic surface where $x' < 0$. By the continuity of the solutions, these regions must be bounded by closed curves on which $x' = 0$ and along which, therefore, the asymptotic surface is tangent to the motion boundary. Further, these bounding curves cannot intersect the earth, by assumption (3).

When the four assumptions are satisfied and a trajectory intersects the earth within the simply connected region mentioned in assumption (2) with $x' > 0$, the trajectory from that point of intersection on, but

excluding that point, will be said to lie in the restricted main cone.

Theorem 1: No half trajectory which lies in the restricted main cone intersects the earth.

Traverse a half trajectory which lies in the restricted main cone, away from its initial point. Since it has $x' > 0$ initially, it must pass through a maximum in x , i.e., $x' = 0$ before it can again intersect the earth. It can only do so by becoming tangent to the motion boundary. In order to reach the motion boundary it must first penetrate the asymptotic surface. Because of the existence and uniqueness theorems, since at such penetration it has $x' > 0$, the asymptotic surface must have $x' < 0$ at the point of penetration. The trajectory, once having so penetrated the asymptotic surface will find itself in a region of (x, λ, λ') space completely bounded by a portion of the motion boundary and a portion of the asymptotic surface everywhere on which $x' < 0$, and which, by assumption (3), does not contain any part of the earth. The trajectory can only leave this region by penetrating the bounding portion of the asymptotic surface again, and must consequently again have $x' > 0$ when it does so. But it will then still be going away from the earth. It is then clear that the theorem is established.

THE PENUMBRA

The penumbra is that range of conditions lying between Störmer's necessary condition and the main cone. As is well known, it consists, in general, of infinitely many allowed and infinitely many forbidden trajectories. Schremp concluded from an inspection of computed trajectories that in any particular direction of observation, at a latitude sufficiently near the equator, the penumbra consists only of forbidden trajectories.

He called this phenomenon the F_0 cutoff. We will now prove this property.

We need the same asymptotic trajectories, the earth, and assumptions as for theorem 1. In addition, we assume

5. The asymptotic surface is tangent to the motion boundary along a closed curve which encircles the asymptotic surface and cannot be shrunk to a point, and which nowhere touches the earth.

This is easy to contrive whenever the first four assumptions are satisfied. Notice that in two of its quadrants the periodic orbit has $x' < 0$. Figure 3 illustrates the argument we now make. It represents a portion of the asymptotic surface slit along the curve $\lambda' = 0$, $\lambda > 0$ and spread out flat. The vertical line represents the periodic orbit. Clearly, that portion of the figure to the left of the periodic orbit, except for other possible regions of $x' < 0$, is established by the property noticed for the periodic orbit together with assumption (3). From Poincare, we know that there are two surfaces asymptotic on the outer side of the periodic orbit. If we choose the one of those that consists of trajectories asymptotic in the infinite future, the portion of Fig. 3 to the right of the periodic orbit is established.

Theorem 2: When assumptions 1-5 hold, any trajectory which intersects the earth with $x' > 0$ in such a point that it does not lie in the restricted main cone will intersect the earth at least once more before going to infinity.

Any trajectory having the initial conditions specified will be entering a region of (x, λ, λ') space completely bounded by segments of the earth, the motion boundary, and the asymptotic surface. At every point on the bounding portions of the asymptotic surface, $x' > 0$. We will call

this region the pocket. The region in which a trajectory in the restricted main cone starts will be called the channel. Any trajectory which goes to infinity must enter the channel. A trajectory which starts in the pocket can do so only by penetrating either the earth or the asymptotic surface at a point where $x' > 0$. In the first case, the theorem is established. In the second case, it must have $x' < 0$ at the time of penetration. It cannot, while still in the channel, acquire a value of $x' > 0$, since no portion of the channel is bounded by the motion boundary. When $x' < 0$, it is going toward the earth, and thus cannot escape to infinity. Consequently, any trajectory which starts in the pocket and finds itself in the channel without having penetrated the earth can only re-enter the pocket or intersect the earth. The theorem is established.

THE SHADOW CONE

Figure 4 illustrates the trajectories with which we are now concerned. Each such trajectory leaves the earth with $x' > 0$, $\lambda' > 0$, passes through a maximum in λ , then one in x , and finally becomes tangent to the earth again. Sometimes one such trajectory issues from a given point, sometimes two do. We call such a trajectory a shadow trajectory. Consider a trajectory with either of the following properties.

1. It leaves the earth with $x' > 0$, $\lambda' > \lambda'_1$ where there issues from the same point only a single shadow trajectory having $\lambda' = \lambda'_1$.
 2. It leaves the earth with $x' > 0$, and either $\lambda' > \lambda'_1$ or $\lambda' < \lambda'_2$. Exactly two shadow trajectories issue from the same point on the earth, one with $\lambda' = \lambda'_1$, the other with $\lambda' = \lambda'_2$.
- Any such trajectory will be said to be in shadow. Schremp assumed that

any trajectory issuing from the earth in such a point that it is in shadow must again intersect the earth before it can proceed to infinity.

Notice first that it is a simple consequence of theorem 1 that no orbits allowed by that theorem can be forbidden by the shadow cone.

In order to understand the shadow cone, we must consider a surface in the (x, λ, λ') space to be called the shadow surface. It is composed entirely of shadow trajectories. Inspired by families of shadow trajectories such as that shown in Fig. 5, we assume the following properties.

3. The most southern¹ member leaves the earth with $x' = 0$, becomes tangent to the motion boundary at $\lambda' = 0$, and then returns to a second tangency with the earth. It has no other extrema in x between the two tangencies with the earth. We designate the point of tangency with the earth by λ_0 .

4. Designate by λ_1 the intersection of the earth with the curve obtained by equating the right member of Eq. (4.a) to zero. At each latitude between λ_0 and λ_1 there issues from the earth a shadow trajectory, having $\lambda' > 0$ at the earth, which passes through a maximum in λ , then one in x , and then becomes tangent to the earth at a latitude greater than λ_0 .

5. There is a latitude, to be called λ_2 , north of which no shadow trajectories issue from the earth. The intersection of the earth, the motion boundary, and $\lambda' = 0$ is north of λ_2 .

6. At every latitude between λ_1 and λ_2 there issue from the earth exactly two shadow orbits, each of which has its tangency north of λ_0 . These issue from the earth with $\lambda' \leq 0$.

7. As one goes from λ_1 to λ_2 , the initial slopes in the (x, λ) space become more and more nearly equal. At λ_2 there is just one shadow trajectory.

Figure 6 shows the shadow surface and that part of the earth bounded by its intersection with the motion boundary. The curves which lie both in these surfaces and in the motion boundary are shown dashed. The shadow surface is shown shaded. The curve DBC is the southernmost shadow orbit which begins and ends at λ_0 . The curve DA consists of the initial points of shadow orbits. The curve AC consists of their tangency points. The curve AB is the locus of their maxima in x , and consequently the shadow surface is tangent to the motion boundary along this curve. On the part of the shadow surface bounded by DBAD, $x' > 0$, on the remainder it is negative.

We require one more surface. It consists of all lines which both pass through the southernmost shadow orbit and intersect the plane $\lambda' = 0$ orthogonally. (A plot in the (x, λ) plane of the southernmost shadow orbit always has DB and CB coincident.) That part of this last surface bounded by its intersection with the motion boundary and the curve DB will be called S_1 . That part bounded by its intersection with BC will be called S_2 . That part bounded by DBCD will be called S_3 .

That region of (x, λ, λ') space completely bounded by portions of the earth, the motion boundary, the shadow surface, and S_1 will be called I. That part bounded completely by portions of the motion boundary, the shadow surface, and S_2 will be called II. That region bounded by part of the earth, the shadow surface, and S_3 will be called III. The segment of surface consisting of $S_1 + S_2 + S_3$ will be called S.

Theorem 3: No trajectory which starts in region I can cross S until it has first crossed the earth.

We establish this result by considering cumulatively various alternatives .

1. No trajectory starting in I can cross S_2 without first crossing S_1 , S_3 , or the earth. To do so it would first have to cross the part of the shadow surface where $x' > 0$. At this crossing, by the uniqueness theorem, it must itself have $x' < 0$. It would then have to cross the part of the shadow surface where $x' < 0$, which it could do only by first changing the sign of its own x' . This it cannot do, since it is in III, no part of which is bounded by the motion boundary.

2. No trajectory starting in I can cross S_3 without first crossing S_1 or the earth. To do so it must first cross the portion of the shadow surface where $x' > 0$, so that when it is in III it has $x' < 0$. But it cannot cross S_3 with $x' < 0$ for the following reason. As can be easily seen by a consideration of the relative slopes of trajectories plotted in the (x, λ) plane, any trajectory crossing S from left to right while having $x' < 0$ must be leaving II, i.e., crossing S_2 , rather than be leaving III via S_3 .

3. No trajectory starting in I can cross S_1 without first crossing the earth. Since the point B in Fig. 6 is in the $\lambda' = 0$ plane, any trajectory crossing S_1 has λ' greater than that at the same (x, λ) point on the $\lambda' > 0$ half of the southernmost shadow orbit. But any trajectory with this property is crossing S_1 to enter I, not to leave it.

The theorem is established.

ON APPLICATIONS

In applying these theorems it is evidently necessary to construct the bounding families and see where they possess the properties assumed. The only families of asymptotic trajectories that we have considered here

exhibit the assumed properties for values of x greater than that corresponding to Schremp's F_0 cutoff, except near the periodic orbit, where the earth's surface cuts through regions with $x' < 0$. At smaller values of x , the assumptions given apparently do not hold. Presumably from an inspection of computed families, one could extend the main cone theorem using techniques like those exploited here. It is evident that this latter case, as well as the just-mentioned small region near the periodic orbit are the situations where the shadow cone theorem has importance.

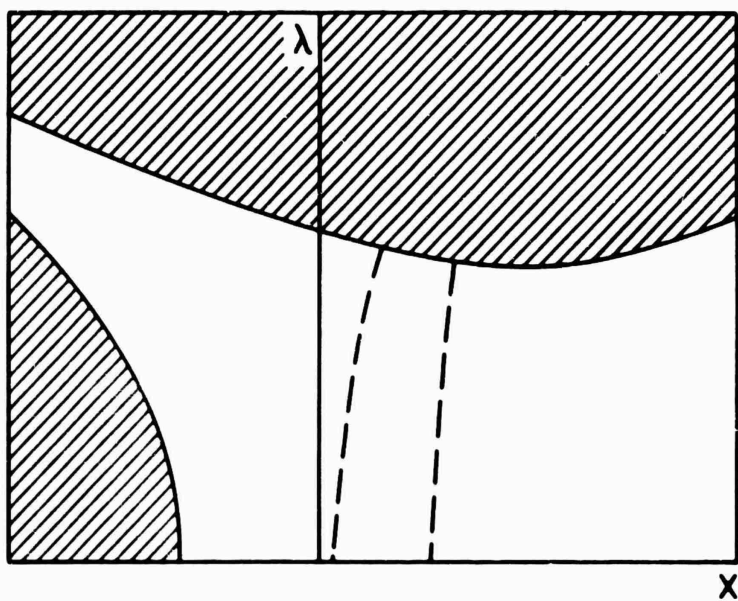


Fig. 1 The shaded regions are forbidden. The solid curves are the intersection of the motion boundary with the plane $\lambda' = 0$. The dashed curves are the projections on the plane $\lambda' = 0$ of the principal orbits. The figure is drawn for $\gamma = 0.8$.

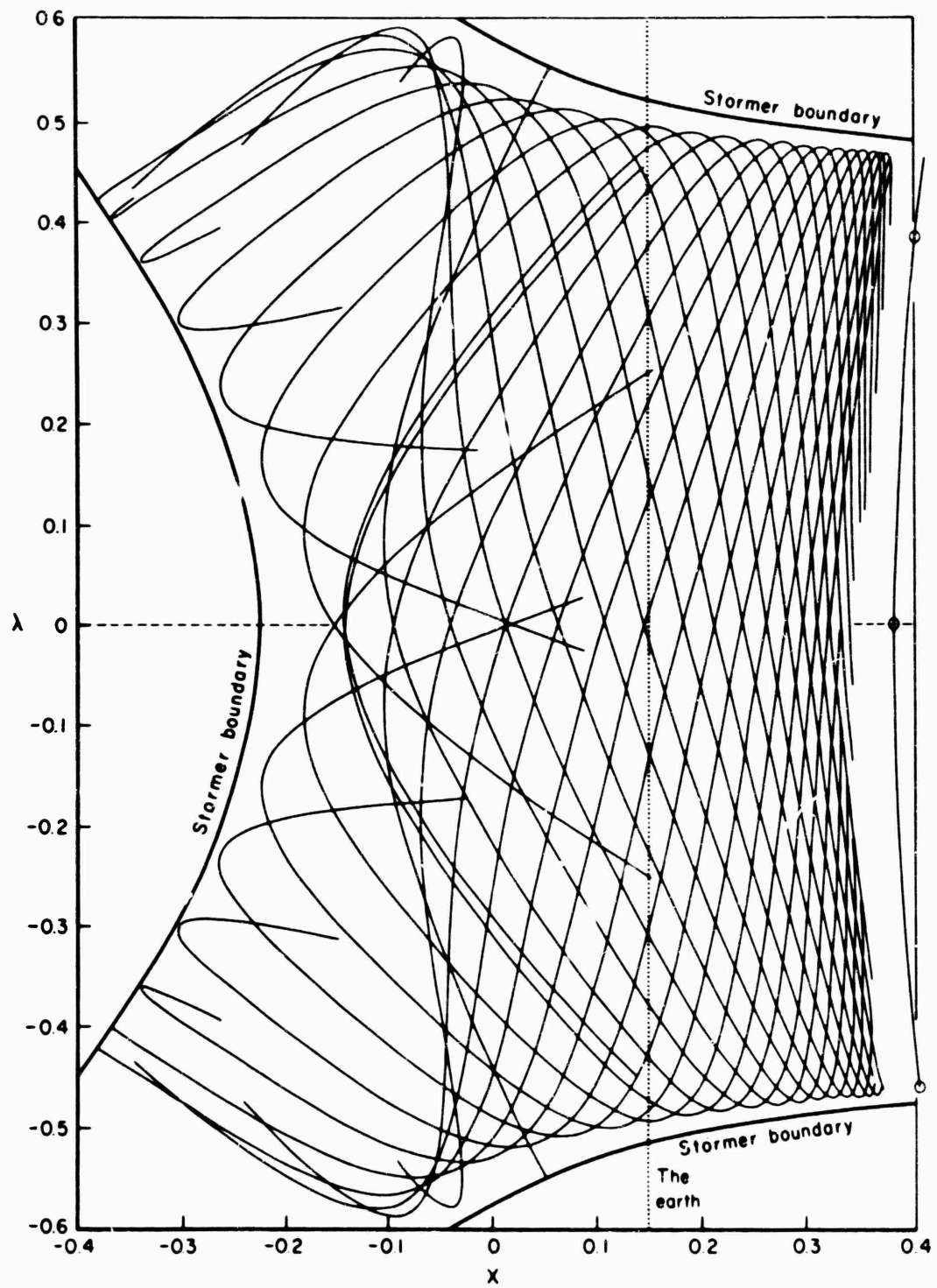


Fig. 2 The inner asymptotic family for $\gamma = 0.85$.

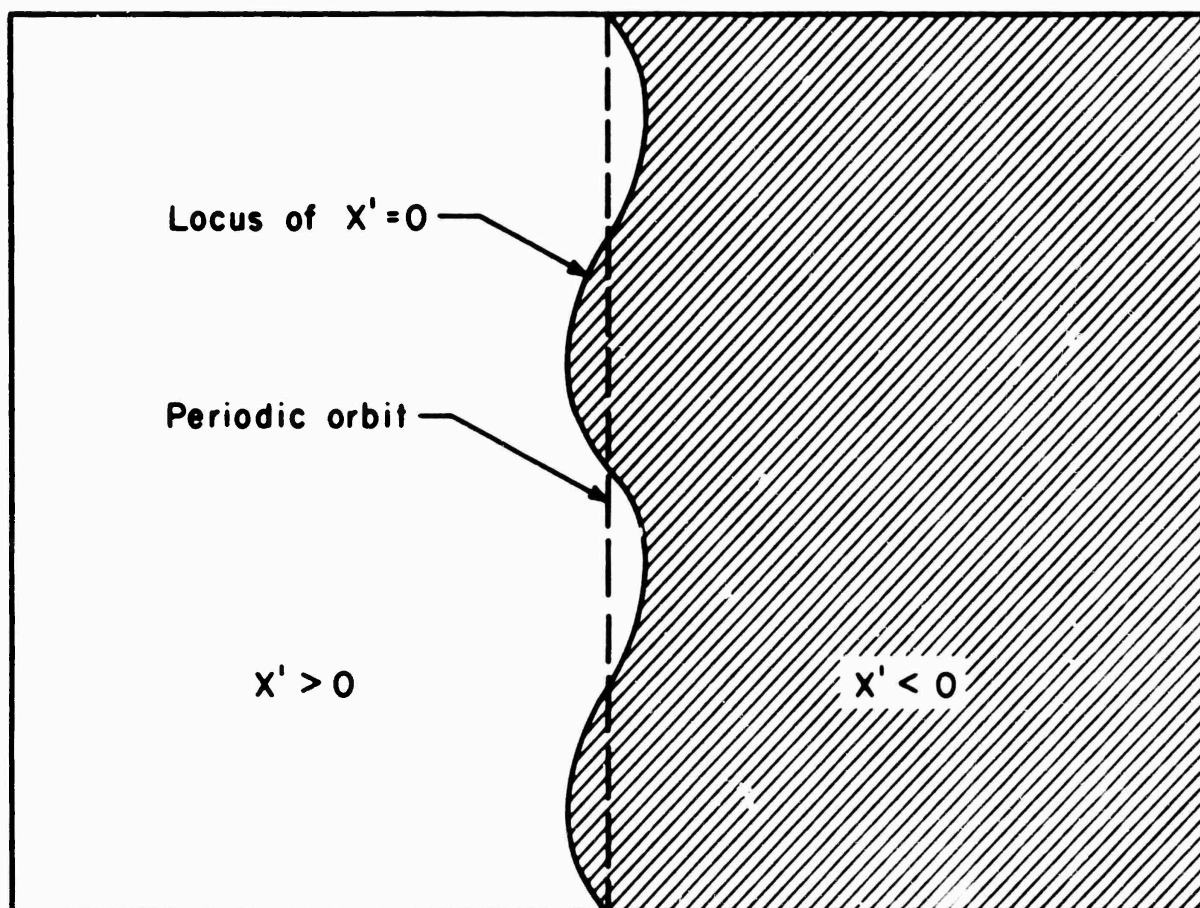


Fig. 3 A representation of an asymptotic surface.

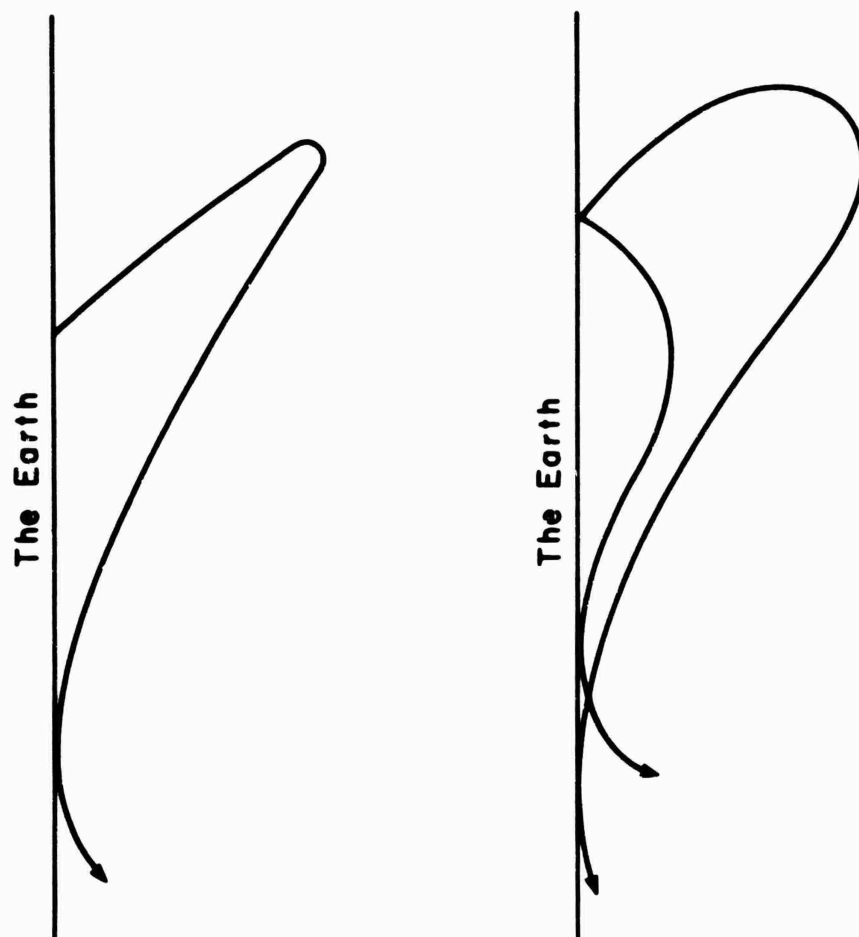


Fig. 4 — Shadow orbits. This figure is schematic and has no quantitative significance.

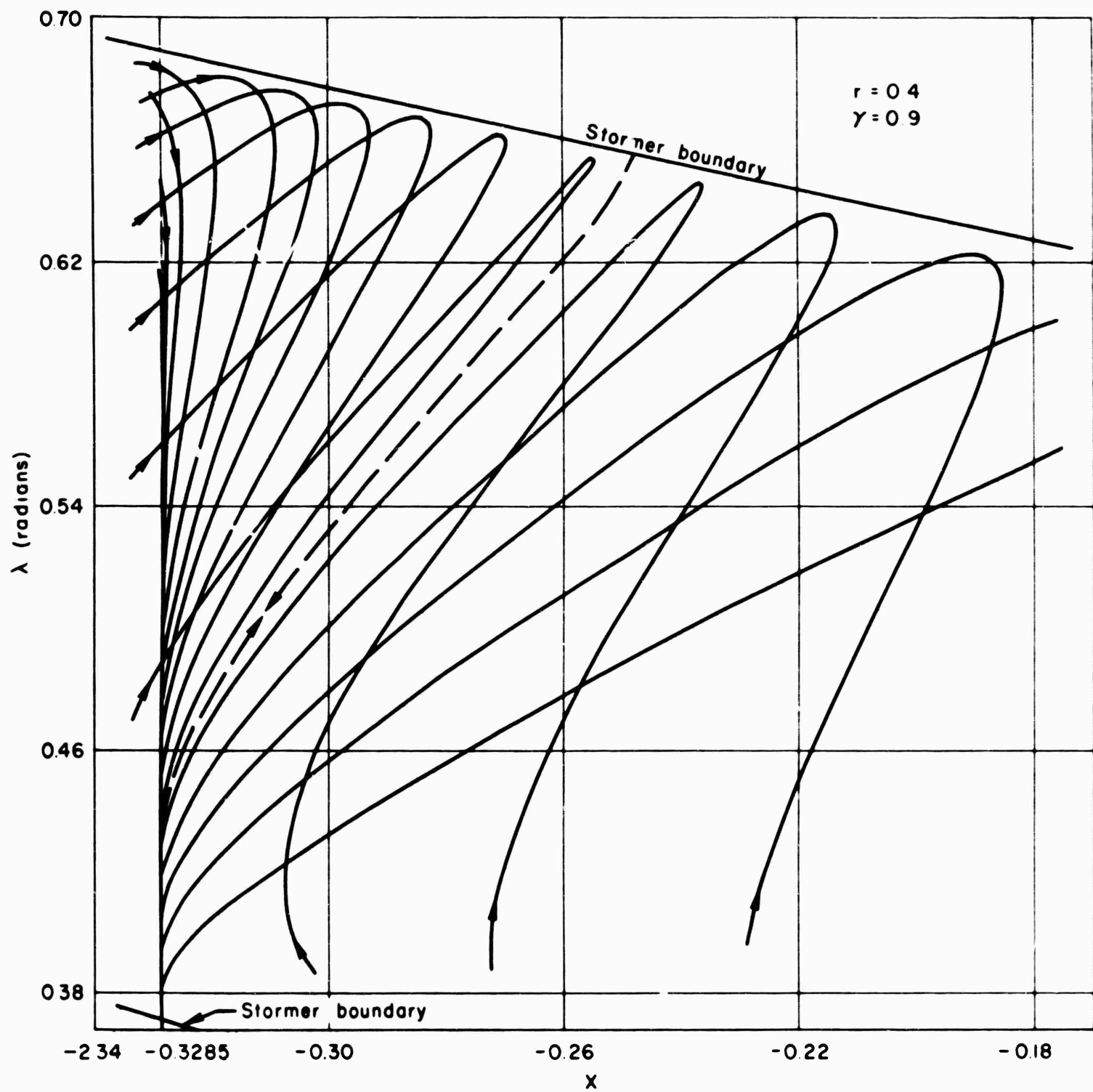


Fig. 5 A family of shadow orbits when the radius of the earth is 0.4 Störmer units and $\gamma = 0.9$.

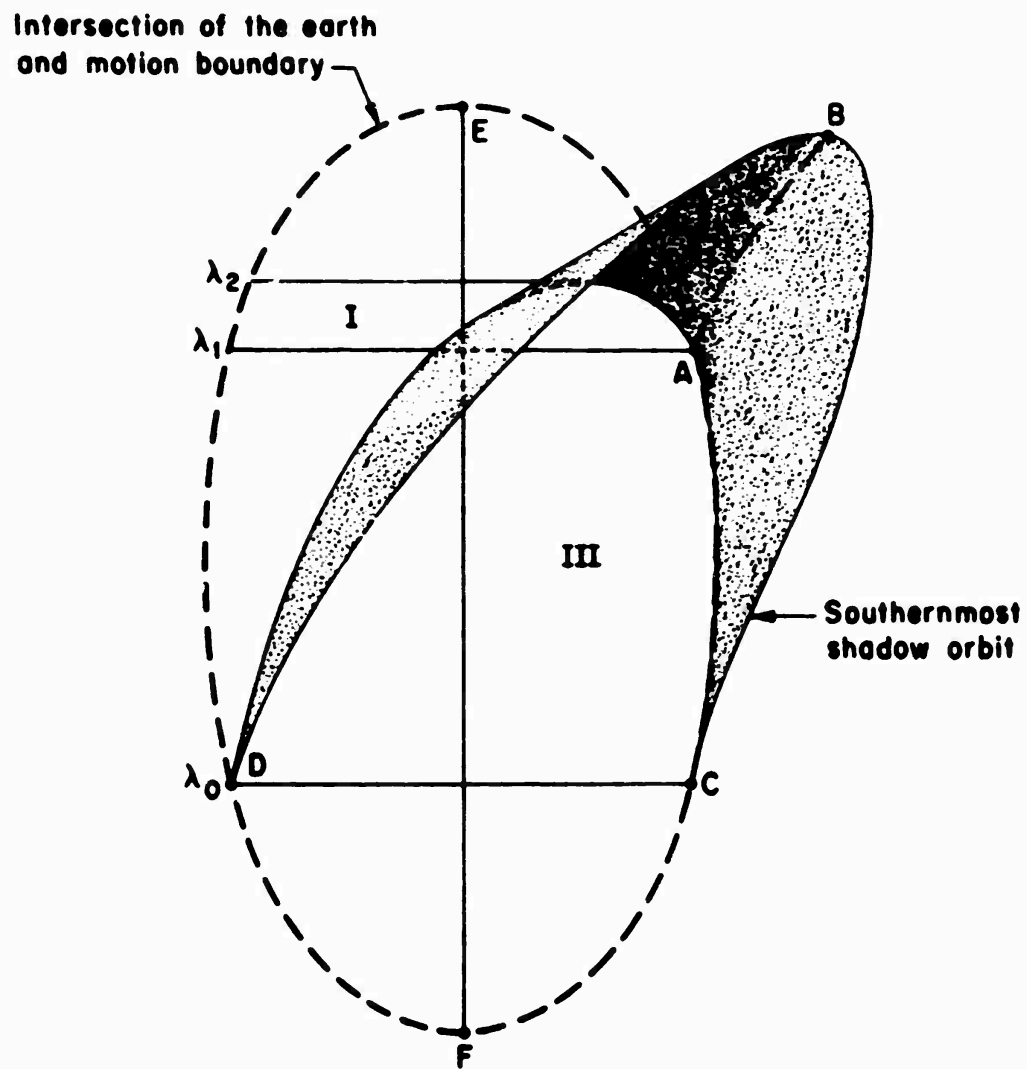


Fig. 6 A schematic representation of a shadow surface
(the shaded surface).

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FOOTNOTES

*On leave from the Department of Physics and Astronomy, State University of Iowa, Iowa City, Iowa, where most of the work was done. The research was supported by NASA Contracts No. NASw-17 at the State University of Iowa and No. NAS5-276 at The RAND Corporation.

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¹The terminology has been adapted, for simplicity, to the case of a family, each member of which has $\lambda' < 0$ at its tangency with the earth.

FIGURE CAPTIONS

- Fig. 1 The shaded regions are forbidden. The solid curves are the intersection of the motion boundary with the plane $\lambda' = 0$. The dashed curves are the projections on the plane $\lambda' = 0$ of the principal orbits. The figure is drawn for $\gamma = 0.8$.
- Fig. 2 The inner asymptotic family for $\gamma = 0.85$.
- Fig. 3 A representation of an asymptotic surface.
- Fig. 4 Shadow orbits. This figure is schematic and has no quantitative significance.
- Fig. 5 A family of shadow orbits when the radius of the earth is 0.4 Störmer units and $\gamma = 0.9$.
- Fig. 6 A schematic representation of a shadow surface (the shaded surface).